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ON A DEFINITION AND FINDING THE GUARANTEED SOLUTION AND GUARANTEED SUBOPTIMAL SOLUTION WITH RESPECT TO THE RESTRICTION CONDITION IN THE BOOLEAN PROGRAMMING PROBLEM WITH A SINGLE RESTRICTION

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Abstract. In the paper a Boolean programming problem with single restriction is considered. The definition of the guaranteed solution and guaranteed subsolution with respect to the coefficients of the restriction condition is given. A method is developed to find the guaranteed suboptimal solution. An example is solved by this method, software is developed and experiments on the different size problems are given.

Keywords: Boolean programming problem, single restriction, guaranteed solution, guaranteed suboptimal solution, dichotomy principle.

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1 Introduction

Consider the following Boolean programming problem with a single restriction

$$\sum_{j=1}^{n} c_j x_j \quad \to max, \tag{1}$$

$$\sum_{j=1}^{n} a_j x_j \quad \le b \ , \tag{2}$$

$$x_j = 0 \vee 1, \quad \left(j = \overline{1,n}\ \right). \tag{3}$$

Not disturbing the generality we assume that here $c_j > 0$, $a_j > 0$, $\left(j = \overline{1,n}\right)$ and $b > 0$ are given integers. Under these conditions problem (1)-(3) is called also a Knapsack problem. To explain the stimulation for consideration of the problem we give here an economic interpretation for problem (1)-(3). Assume the coefficients a_j, $\left(j = \overline{1,n}\ \right)$ stand for the expenses for the j-th object, c_j, $\left(j = \overline{1,n}\right)$ denote the benefit from the use of the j–th object, b is the general resource.

The problem is: to choose the objects for which the general investment does not exceed b and the benefit would be maximum.

Note that problem (1)-(3) is from the NP-integer class. Other words do not exist the methods with polynomial complicity for finding its optimal solution. But some "branches and boundaries", "dynamic programming", "combinatory type" methods have been developed for the solution of the problems with less number of variables (Korbut & Finkelstein, 1969; Kovalev, 2003;

Emelichev & Komlik, 1981; Martello & Toth, 1990; Pisinger, 1997; Veliyev & Mamedov, 1981). Sometimes it is expedient to construct approximate (suboptimal) solution to these problems that is close enough to the exact one.

In this paper we aim to find a solution to problem (1)-(3) that gives value to functional (1) which is greater than given one using the small variation of the coefficients of the restriction conditions. That is why this solution is called a guaranteed with respect to the coefficients of the restriction condition solution.

Note that the definition of the guaranteed solution and guaranteed suboptimal solution with respect to the right hand side b of the restriction condition or coefficients $c_j, \left(j = \overline{1,n}\right)$ of functional (1) was introduced in (Mamedov & Mamedov, 2012, 2014, 2015, 2018a,b). In addition the procedures of finding the optimal solutions of the integer and Boolean programming problems are investigated in (Bukhtoyarov & Emelichev, 2019; Emelichev & Mychkov, 2016; Haldik, 2017; Emelichev & Nikulin, 2018; Mostafaee et al., 2016).

Here we assume the possibility to find the guaranteed solution not changing the resource b and coefficients $c_j, \left(j = \overline{1,n}\right)$ by variation of the expenses a_j, $\left(j = \overline{1,n}\right)$ in some interval.

It should be noted that such problems arise usually in the enterprises with economic difficulties.

2 Formulation of the problem

Suppose that problem (1)-(3) had been solved by some method and its optimal

$$X^* = (x_1^*, x_2^*, \ \dots, \ x_n^*)$$

or suboptimal solution $X^s = (x_1^s, x_2^s, \ \dots, \ x_n^s)$ had been found. Then the value of (1) will be

$$f^* = \sum_{j=1}^{n} c_j x_j^* \ \text{ or } f^s = \sum_{j=1}^{n} c_j x_j^s \ .$$

Let we aim to get more benefit than the found values f^* or f^s. Other words we want to find the solutions $X^z = (x_1^z, x_2^z, \dots, x_n^z)$ or $X^{zs} = (x_1^{zs}, x_2^{zs}, \dots, x_n^{zs})$ of problem (1)-(3) such that the correspond values of functional (1)

$$f^z = \sum_{j=1}^{n} c_j x_j^z \ or \ f^{zs} = \sum_{j=1}^{n} c_j x_j^{zs}$$

are guaranteed to be greater than the numbers f^* or f^s. This means that the conditions $f^z \geq f^* + \Delta^*$ or $f^{zs} \geq f^s + \Delta^s$ hold. Particularly one may set $\Delta^* = \left[f^* \frac{p}{100}\right]$ or $\Delta^s = \left[f^s \frac{p}{100}\right]$. Here the number p is a given increasing percentage of the numbers (benefits) f^* or f^s, and $[z]$ stands for the integer part of the number z.

For this purpose we should minimally vary the expenses $a_j, \left(j = \overline{1,n}\right)$ in the given interval $[0, \alpha_j]$, $\left(j = \overline{1,n}\right)$.

Then we get the following new problem

$$\delta_j \to min, \left(j = \overline{1,n}\right), \tag{4}$$

$$\sum_{j=1}^{n} c_j x_j \geq f^* + \Delta^*, \tag{5}$$

$$\sum_{j=1}^{n} (a_j - \delta_j) x_j \leq b, \tag{6}$$

$$0 \le \delta_j \le \ \alpha_j, \ \left(j = \overline{1,n}\right) \ are \ integers, \tag{7}$$

$$x_j = 0 \vee 1, \ \left(j = \overline{1,n}\right). \tag{8}$$

Here $c_j, a_j, \left(j = \overline{1,n}\right), b, f^*$ and Δ^* are given integers, x_j and δ_j, $\left(j = \overline{1,n}\right)$ are unknown. After the solving process the number a_j should decrease by δ_j.

Note that problem (4)-(8) is multicriteria nonlinear Boolean programming problem (nonlinearity is expressed by the product $\delta_j \bullet x_j$ in condition (6)).

Thus problem (4)-(8) is also from NP-class. Therefore setting f^s instead of f^* and Δ^s instead of Δ^* in (4)-(8) we get the similar problem

$$\delta_j \to min, \ \left(j = \overline{1,n}\right), \tag{9}$$

$$\sum_{j=1}^{n} c_j x_j \ge f^s + \Delta^s, \tag{10}$$

$$\sum_{j=1}^{n} (a_j - \delta_j) x_j \le b, \tag{11}$$

$$0 \le \delta_j \le \alpha_j, \ \left(j = \overline{1,n}\right) are \ intergers, \tag{12}$$

$$x_j = 0 \vee 1, \left(j = \overline{1,n}\right). \tag{13}$$

Here $c_j > 0$, $a_j > 0, \alpha_j \ge 0, \left(j = \overline{1,n}\right), b > 0$ and $\Delta^s > 0$ are given integers, x_j and δ_j, $\left(j = \overline{1,n}\right)$ are searched variables. It is clear that the natural conditions $\alpha_j < a_j$, $\left(j = \overline{1,n}\right)$ should be satisfied.

3 Theoretical justification of the method

Here we give some definitions necessary for developing the solution method of problem (9)-(13).

Definition 1. *As an admissible solution of problem* (4)-(8) *we consider* n *dimensional vector* $X = (x_1, x_2, \ldots, x_n)$ *that satisfies conditions* (5)-(8) *for each fixed* δ_j, $\left(j = \overline{1,n}\right)$.

Definition 2. *The admissible solution* $X^z = (x_1^z, x_2^z, \ldots, x_n^z)$ *is called a guaranteed with respect to the coefficients of the restriction condition solution of problem* (4)-(8) *that provides minimal values to* $\delta_j, \left(j = \overline{1,n}\right)$.

Note that since problem (4)-(8) belongs to the class of multicriteria Boolean programming problems, it is also from NP-integer class. So there does not exist methods with polynomial complicity for its solving. From other side problem (4)-(8) is nonlinear (due to the products $\delta_j \bullet x_j$, $\left(j = \overline{1,n}\right)$ in inequalities (6)). So its solving requires nonreal calculation time. That is why it is expedient from both theoretical and practical points of view to define its guaranteed suboptimal solution and develop methods for its finding.

Definition 3. *Guaranteed suboptimal solution for problem* (9)-(13) *is called the vector* $X^{zs} = (x_1^{zs}, x_2^{zs}, \ldots, x_n^{zs})$ *that gives minimal values to* δ_j, $\left(j = \overline{1,n}\right)$ *subject to conditions* (10)-(13).

Note that for problem (1)-(3) the definition of the guaranteed with respect to the number b in the right hand side of restriction (2) and coefficients c_j, $\left(j=\overline{1,n}\right)$ of functional (1) was given by author in the works (Mamedov & Mamedov, 2015, 2018a,b). There the corresponding algorithms also were developed.

Here in differ from those works we assume that the resources b and the values c_j, $\left(j=\overline{1,n}\right)$ are constant a_j, $\left(j=\overline{1,n}\right)$ but the expenses should be decreased minimally in the intervals $[0,\alpha_j]$, $\left(j=\overline{1,n}\right)$. For this purpose we offer the solving process for the problem of finding of the guaranteed with respect to the coefficients of the restriction conditions suboptimal solution of problem (9)-(13): first we find by some known method any suboptimal solution $X^{s_0}=(x_1^{s_0},x_2^{s_0},\dots,x_n^{s_0})$ of problem (1)-(3), corresponding value $f^{s_0}=\sum_{j=1}^{n}c_jx_j^{s_0}$ of functional (1) and the number $\Delta^s=\left[f^{s_0}\frac{p}{100}\right]$. Here $[z]$ stands for the integer part of the number z and p is the increasing percentage of the number f^{s_0}. Then we set $f^s=f^{s_0}$ in condition (10) and get model (9)-(13). Our aim is to find such minimal values for δ_j, $\left(j=\overline{1,n}\right)$ in the intervals $[0,\alpha_j]$, $\left(j=\overline{1,n}\right)$ that provide fulfilment of conditions (10)-(13) in problem (9)-(13). So taking $\delta_j:=\alpha_j$, $\left(j=\overline{1,n}\right)$ and setting $a_j':=a_j, \left(j=\overline{1,n}\right)$ we remember them. Then take $a_j:=a_j'-\delta_j, \left(j=\overline{1,n}\right)$. As a result we get problem (1)-(3) the suboptimal solution $X^0=(x_1^0,x_2^0,\dots,x_n^0)$ for which may be found by known method and the value

$$f^0=\sum_{j=1}^{n}c_jx_j^0$$

may be calculated. It is clear that $f^0>f^s+\Delta^s$ since the coefficients $a_j, \left(j=\overline{1,n}\right)$ have been decreased maximally. To minimize the values $\delta_j, \left(j=\overline{1,n}\right)$ we use dichotomy principle and define new quantities $\delta_j, \left(j=\overline{1,n}\right)$: setting $\gamma_j:=0, \beta_j:=\delta_j, t_j:=\gamma_j, z_j:=\beta_j, \delta_j:=\left[\frac{\gamma_j+\beta_j}{2}\right]$, $\left(j=\overline{1,n}\right)$ find $a_j:=a_j'-\delta_j, \left(j=\overline{1,n}\right)$. In result we get new intermediate problem (1)-(3) the suboptimal solution $X^1=(x_1^1,x_2^1,\dots,x_n^1)$ of which and corresponding value

$$f^1=\sum_{j=1}^{n}c_jx_j^1$$

of functional (1) may be found.

Here we get two cases:

Case 1. $f^1\geq f^s+\Delta^s$.

Case 2. $f^1<f^s+\Delta^s$.

In the first case we remember $f^{zs}:=f^1$, $X^{zs}:=X^1$. Then to minimize the values δ_j, $\left(j=\overline{1,n}\right)$ we set $\gamma_j:=t_j, \beta_j:=\delta_j, z_j:=\beta_j, \overline{\delta}_j:=\delta_j$, and $\delta_j:=\left[\frac{\gamma_j+\beta_j}{2}\right]$, $\left(j=\overline{1,n}\right)$ and take $a_j:=a_j'-\delta_j$, $\left(j=\overline{1,n}\right)$.

In the second case taking $\gamma_j:=\delta_j, \beta_j:=z_j, t_j:=\gamma_j$ *and* $\delta_j:=\left[\frac{\gamma_j+\beta_j}{2}\right]$, $\left(j=\overline{1,n}\right)$ we calculate $a_j:=a_j'-\delta_j, \left(j=\overline{1,n}\right)$.

It is clear that only one of these cases may occur in each iteration.

So we obtain new problem (1)-(3). Continuing this process in some k-th step we find suboptimal solution $X^k=(x_1^k,x_2^k,\dots,x_n^k)$ and corresponding value $f^k=\sum_{j=1}^{n}c_jx_j^k$ of functional (1). It is clear that the process can be continued till satisfying the relations $\beta_j-\gamma_j\leq 1, \left(j=\overline{1,n}\right)$ for all numbers $j, \left(j=\overline{1,n}\right)$. Other words after this the operation of dividing by 2 will give the same result. Note that if in any l -th step of iteration we get $f^l\geq f^s+\Delta^s$ $(1\leq l\leq k)$ then we should remember $f^{zs}:=f^l$, $X^{zs}:=X^l$ and $\overline{\delta}_j=\delta_j, \left(j=\overline{1,n}\right)$.

If in some k -th step we have $\beta_j-\gamma_j\leq 1, \left(j=\overline{1,n}\right)$ then the vector $X^{zs}=(x_1^{zs},x_2^{zs},\dots,x_n^{zs})$ will be guaranteed suboptimal solution and the value f^{zs} - searched guaranteed value of function (1).

Example. Now we apply the above proposed method to the solving of the following problem

$$15x_1 + 8x_2 + 12x_3 + 20x_4 + 17x_5 + 14x_6 + 6x_7 + 4x_8 + 5x_9 + 2x_{10} \to max, \tag{14}$$

$$10x_1 + 6x_2 + 10x_3 + 16x_4 + 14x_5 + 11x_6 + 4x_7 + 3x_8 + 4x_9 + 2x_{10} \le 30. \tag{15}$$

$$x_j = 0 \vee 1, \left(j = \overline{1,10}\right) \tag{16}$$

Let the values $\delta_j, \left(j = \overline{1,10}\right)$ should vary in the following intervals

$$\begin{gathered} \delta_1 \in [0,8], \delta_2 \in [0,4], \delta_3 \in [0,7], \delta_4 \in [0,13], \delta_5 \in [0,12], \delta_6 \in [0,9], \\ \delta_7 \in [0,3], \delta_8 \in [0,2], \delta_9 \in [0,3], \delta_5 \in [0,1]. \end{gathered} \tag{17}$$

Here

$$\delta_j = (8,4,7,13,12,9,3,2,3,1).$$

Suboptimal solution of problem (14)-(16) is found by known method and is

$$X_0 = (1,1,0,0,0,0,1,1,1,1).$$

Then function (14) takes the corresponding value $f^0 = 40$.

If we want to get the value that is greater than f^0 by $p = 20\%$ then

$$\Delta^S = \left[f^0 \frac{p}{100}\right] = \left[40 \bullet \frac{20}{100}\right] = 8.$$

Thus $f^0 + \Delta^S = 48$. Then problem (9)-(13) turns to

$$\delta_j \to min, \left(j = \overline{1,10}\right) \tag{18}$$

$$15x_1 + 8x_2 + 12x_3 + 20x_4 + 17x_5 + 14x_6 + 6x_7 + 4x_8 + 5x_9 + 2x_{10} \ge 48, \tag{19}$$

$$\begin{gathered} (10-\delta_1)x_1 + (6-\delta_2)x_2 + (10-\delta_3)x_3 + (16-\delta_4)x_4 + (14-\delta_5)x_5 + \\ + (11-\delta_6)x_6 + (4-\delta_7)x_7 + (3-\delta_8)x_8 + (4-\delta_9)x_9 + (2-\delta_{10})x_{10} \le 30, \end{gathered} \tag{20}$$

$$\begin{gathered} \delta_1 \in [0,8], \delta_2 \in [0,4], \delta_3 \in [0,7], \delta_4 \in [0,13], \delta_5 \in [0,12], \delta_6 \in [0,9], \\ \delta_7 \in [0,3], \delta_8 \in [0,2], \delta_9 \in [0,3], \delta_5 \in [0,1], \end{gathered} \tag{21}$$

$$x_j = 0 \vee 1, \left(j = \overline{1,10}\right). \tag{22}$$

We set $\delta_j = (8,4,7,13,12,9,3,2,3,1)$. Then the coefficients $a_j, \left(j = \overline{1,10}\right)$ of condition (15) take the values $a_j = (2,2,3,3,2,2,1,1,1,1)$. Therefore we get $f^0 + \Delta^S = 48$ for problem (14)-(16).

Thus we get intermediate problem (14)-(16). Solution of this problem is

$$X^1 = (1,1,1,1,1,1,1,1,1,1)$$

and $f^1 = 103 > 48$. It is natural that we should decrease the values $\delta_j, \left(j = \overline{1,10}\right)$. Using dichotomy principle we get

$$\delta_1 \in [0,4], \delta_2 \in [0,2], \delta_3 \in [0,3], \delta_4 \in [0,6], \delta_5 \in [0,6], \delta_6 \in [0,4],$$

$$\delta_7 \in [0,1]\,, \delta_8 \in [0,1]\,, \delta_9 \in [0,1]\,, \delta_5 \in [0,1]\,.$$

Since $f^1 = 103 > 48$ we should choose the values $\delta_j, \left(j = \overline{1,10}\right)$ as $\delta_j = (4,3,3,6,6,4,1,1,1,1)\,.$ Then the coefficients of (15) will be as follows $a_j = (6,4,7,10,8,7,3,2,3,1)\,.$ Then solving the obtained intermediate problem (14)-(16) we find the solution $X^2 = (0,1,0,1,1,1,0,0,0,1)$ and corresponding value of the function $f^2 = 61 > 48.$

It is clear that since $f^2 > 48$ we should decrease the values $\delta_j, \left(j = \overline{1,10}\right)$ using dichotomy principle. Continuing the process after the third step we get

$$\min_j \delta_j = (2,1,1,3,3,2,0,0,0,0)\,.$$

Corresponding solution $X^S = (1,1,0,0,1,0,1,0,0,1)$ and value for the function $f^{zs} = 48$. Finally we get

$$\sum_{j=1}^{n} a_j \ = 83, \quad \sum_{j=1}^{n} a'_j = 68\ , \quad \Delta = 83 - 68 = 15, \quad f^0 = 40, \quad p = 20\%,$$

$$\Delta^s = 8, \quad f^0 + \ \Delta^s = 40 + 8 = 48, \quad f^0 - \ f^0 = 48 - 40 = 8.$$

Thus we found the solution that guarantees increasing of the cost function by minimum 8 units under decreasing the coefficients of the restriction conditions by 15 units.

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